Infinity Computer: methodology, patents, prototypes and applications

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Outline

Recent awards and distinctions for global optimization

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- EUROPT Fellow, 2016
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New computational paradigms are the future of computations

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- Current attention of the scientific and industrial communities is dedicated to new computational paradigms (all kinds of distributed computing, quantum computing, Bio computing, etc.) and areas of their possible applications.
- However, **there do not exist on the market unconventional alternatives ready for a broad implementation**. For instance, a powerful general purpose quantum computer has not been constructed yet (D-Wave is not general purpose).

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The Infinity Computer is a radical innovation

- **Limitations of traditional computers:** They work with finite numbers only. Numerical computations with different infinities and infinitesimals were impossible so far.
- **The Infinity Computer is a radical innovation** since it is able to execute numerical (not symbolic) calculations with finite, infinite and infinitesimal numbers in a unique framework opening so **a new era in the whole world of computing**.
- **The Infinity Computer drastically increases the accuracy of computations** by substituting qualitative descriptions of the type "a number tends to zero" by a variety of precise infinitesimal numbers; **it avoids indeterminate forms** (e.g., $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, etc.); **it eliminates all kinds of divergences** since it becomes possible to work directly with actual infinite and infinitesimal numbers.

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Intellectual property and prototypes

- **Patents.** Sergeyev, Ya.D., Computer system for storing infinite, infinitesimal, and finite quantities and executing arithmetical operations with them

  USA patent 7,860,914, EU patent 1728149, RF Patent 2395111.

  The patents are written without any specification of the type of hardware implementing the Infinity Computer.

- European trademark *Grossone*, reg.no. 005294764.
- **There is a working software prototype** of the Infinity Computer and the Infinity Calculator using the technology.
- The following websites provide a wealth of information about the Infinity Computer, books, scholarly articles, etc.: **http://www.theinfinitycomputer.com** **http://www.grossone.com**

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Value proposition

- **Market: is the entire market of digital processors** including microprocessors, devices for digital elaboration of signals, scientific calculators, etc.
- **Radical innovation:** The Infinity Computer offers to consumers a completely new way of computations giving the possibility to substitute old weak concepts and computational tools by the new ones offering both an **unprecedented enlargement of computable objects including infinite and infinitesimal numbers and completely new horizons with respect to the accuracy of computations**.
- **Strategic advantage:** In the modern world the power of computation can cause victory or defeat of a firm or a country. The Infinity Computer with its supercomputing capabilities gives a strategic advantage for its owners.

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Scientific prizes, plenary lectures, and mass confirm that **we face a real breakthrough**

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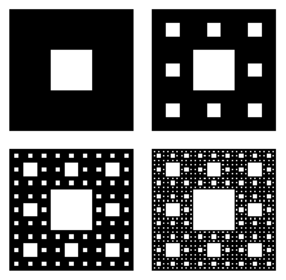
'Infinity Computer' Calculates Area Of Sierpinski Carpet Exactly

Mathematicians have never been comfortable handling infinities, such as those that crop up in the area of a Sierpinski carpet. But an entirely new type of mathematics looks set to by-pass the problem

By KFC



Traditional computers work with finite numbers only

- Arithmetics developed for infinite numbers are quite different with respect to the finite arithmetic we are used to deal with.
  - Presence of undetermined forms;
  - Infinite numbers often are represented as infinite sequences of finite numbers;
  - Technical problems: e.g., how to store an infinite number in a finite memory of a computer?
- These crucial difficulties did not allow people to construct computers that would be able to work with infinite and infinitesimal numbers **numerically** and **in the same manner** as we are used to do with finite numbers.
- Our goal was to construct such a computer and we have succeeded in this.

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Numbers and numerals

- A *numeral* is a symbol (or a group of symbols) that represents a *number*. The difference between numbers and numerals is the same as the difference between words and the things they refer to.
- **A number is a concept that a numeral expresses.**
- A numeral can be written down, copied, erased, and so on and a number cannot. The same number can be represented by different numerals. For example, the symbols '12', 'twelve', and 'XII' are different numerals, but they all represent the same number.
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Numeral systems used to express finite numbers

- Different numeral systems can express different sets of numbers and they can be more or less suitable for executing arithmetical operations.
- Even the powerful positional system is not able to express by a finite number of symbols, e.g., the number π and this special numeral, π, is deliberately introduced to express the desired quantity.
- There exist many numeral systems that are weaker than the positional one. For instance, Roman numeral system is not able to express zero and negative numbers and such an expression as III – V is an indeterminate form in this numeral system.

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Primitive numeral systems

- People from a primitive Amazon tribe of Pirahã (see P. Gordon, *Science*, October 2004) use only numerals I and II and are able to see only numbers 1 and 2. They do not suspect about existence of other natural numbers.
- All quantities larger than II are just 'many' and such operations as $I + II$ and $II + II$ give the same result, i.e., 'many'. Note that this happens not because $I + II = II + II$ but due to weakness of this primitive numeral system.
- This weakness leads also to such results as

$$\text{`many'} + 1 = \text{`many'}, \qquad \text{`many'} + 2 = \text{`many'}$$

- Which are very familiar to us in the context of our views on infinity:

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Important observation

Our difficulty in working with infinity is not connected to the nature of infinity but is a result of inadequate numeral systems used to express infinite numbers.

Counting large quantities in practice: the granary example

- Imagine that we are in a granary and the owner asks us to count how much grain he has inside it. Of course, nobody can count the grain seed by seed and the problem: 'How to count such an enormous number of objects?' arises.
- In order to overcome this difficulty, people take sacks, fill them in with seeds, and count the number of sacks. **It is important that nobody counts the number of seeds in a sack.**
- If the granary is huge and it becomes difficult to count the sacks, then trucks or even big train waggons are used.
- At the end of the counting we obtain a result in the following form: the granary contains 17 waggons, 12 trucks, 15 sacks, and 54 seeds of grain.

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- Imagine that we are in a granary and the owner asks us to count how much grain he has inside it. Of course, nobody can count the grain seed by seed and the problem: 'How to count such an enormous number of objects?' arises.
- In order to overcome this difficulty, people take sacks, fill them in with seeds, and count the number of sacks. **It is important that nobody counts the number of seeds in a sack.**
- If the granary is huge and it becomes difficult to count the sacks, then trucks or even big train waggons are used.
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The first key observation

- In the example it is necessary to count large quantities. They are finite but it is impossible to count them using elementary units of measure u_0 – seeds – because the quantities expressed in these units would be too large. Therefore, people are forced to behave as if the quantities were infinite.
- To solve the problem of 'infinite' quantities, *new units of measure are introduced*: u_1 – sacks, u_2 – trucks, u_3 – train waggons, etc.
- Thus, it was impossible to formulate the answer in the terms of elementary units but it is perfectly expressible in the newly introduced units of measure.

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The second key observation

- It is not known how many units u_i there are in the unit u_{i+1}. **We do not count how many seeds are in a sack, we just complete the sack.**
- Thus, we know that all the units u_{i+1} contain a certain number K_i of units u_i but this number, K_i, is unknown. Naturally, it is supposed that K_i is the same for all instances of the units. This assumption signifies that all sacks contain the same number of seeds, all trucks – the same number of sacks, and all train waggons – the same number of trucks.

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A new numeral to express a new infinite unit

Important observation

The new infinite unit expressed by the numeral ① and called grossone™ is introduced as the number of elements of the set, $\mathbb{N}$, of natural numbers. It is defined through its properties postulated by the Infinite Unit Axiom.

This axiom is added to axioms for real numbers similarly to addition of the axiom determining zero to axioms of natural numbers when integer numbers are introduced.

Past – Present – Future

Piraha: using 1, 2, many

Today: using ∞

Tomorrow: using ①

Measuring infinite sets using ①-based numerals and the traditional cardinalities $\aleph_0$ and $\mathcal{C}$

Description of sets	Cardinality	# of elements
the set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$	countable, $\aleph_0$	①
$\mathbb{N} \setminus \{3, 5, 10, 23, 114\}$	countable, $\aleph_0$	①-5
the set of even numbers $\mathbb{E}$ (the set of odd numbers $\mathbb{O}$)	countable, $\aleph_0$	$\frac{①}{2}$
the set of integers $\mathbb{Z}$	countable, $\aleph_0$	2①+1
$\mathbb{Z} \setminus \{0\}$	countable, $\aleph_0$	2①
squares of natural numbers $\mathbb{G} = \{x : x = n^2, x \in \mathbb{N}, n \in \mathbb{N}\}$	countable, $\aleph_0$	$\lfloor \sqrt{①} \rfloor$
pairs of natural numbers $\mathbb{P} = \{(p, q) : p \in \mathbb{N}, q \in \mathbb{N}\}$	countable, $\aleph_0$	$①^2$
the set of numerals $\mathbb{Q} = \{0, -\frac{p}{q}, \frac{p}{q} : p \in \mathbb{N}, q \in \mathbb{N}\}$	countable, $\aleph_0$	$2①^2 + 1$
numbers $x \in [0, 1)$ expressible in the binary numeral system	continuum, $\mathcal{C}$	$2^{①}$
numbers $x \in [0, 1]$ expressible in the binary numeral system	continuum, $\mathcal{C}$	$2^{①} + 1$
numbers $x \in [0, 1)$ expressible in the decimal numeral system	continuum, $\mathcal{C}$	$10^{①}$
numbers $x \in [0, 2)$ expressible in the decimal numeral system	continuum, $\mathcal{C}$	$2 \cdot 10^{①}$

Extended natural numbers

- Added to the axioms of natural numbers the Infinite Unit Axiom defines the set of *extended natural numbers* indicated as $\widehat{\mathbb{N}}$ and including $\mathbb{N}$ as a proper subset

$$\widehat{\mathbb{N}} = \{\underbrace{1, 2, \ldots, ① - 1, ①}_{\mathbb{N}}, ① + 1, \ldots, ①^2 - 1, ①^2, ①^2 + 1, \ldots\},$$

where

$$1 < 2 < \ldots < ① - 1 < ① < ① + 1 <$$
$$\ldots < ①^2 - 1 < ①^2 < ①^2 + 1 < \ldots$$

Functions and their derivatives

- It becomes possible to calculate functions and their derivatives assuming finite, infinite, or infinitesimal values at finite, infinite, or infinitesimal points and derivatives are calculated without usage of limits.
- For example, the function $f(x) = x^2$ has the first derivative $f'(x) = 2x$.
- For infinite $x =$ ① we obtain infinite values $f(①) = ①^2$ and $f'(①) = 2①$.
- For infinitesimal $x = ①^{-1}$ we have infinitesimal values $f(①^{-1}) = ①^{-2}$ and $f'(①^{-1}) = 2①^{-1}$.

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Functions and their derivatives

- We can also work with functions having formulae including infinite and infinitesimal numbers.
- For example, the function $f(x) = \frac{1}{①}x^2 + ①x$ has the first derivative $f'(x) = \frac{2}{①}x + ①$.
- For infinite $x = ①$ we obtain infinite values $f(①) = ① + ①^2$ and $f'(①) = 2 + ①$.
- For infinitesimal $x = ①^{-1}$ we have $f(①^{-1}) = ①^{-3} + 1$ and $f'(①^{-1}) = 2①^{-2} + ①$.

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Improper integrals

- In order to calculate improper integrals it is necessary to define them in a more precise way. For example, in the following improper integral

$$\int_0^{\infty} x^2 dx$$

  it is necessary to define its limits of integration explicitly.

- Then, two different infinite numbers ① and $①^2$ used instead of ∞ give us two different integrals

$$\int_0^{①} x^2 dx = \frac{1}{3}①^3, \qquad \int_0^{①^2} x^2 dx = \frac{1}{3}①^6.$$

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- Moreover, it becomes possible to calculate integrals where both endpoints of the interval of integration are infinite.
- In this example the result is infinite

$$\int_{①}^{①^2} x^2 dx = \frac{1}{3}①^6 - \frac{1}{3}①^3.$$

- Here the result has finite and infinitesimal parts

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Positional system with infinite radix

- In order to represent a number C, we subdivide it into groups corresponding to powers of ①:

  $$C = c_{p_m}①^{p_m} + \ldots + c_{p_1}①^{p_1} + c_{p_0}①^{p_0} + c_{p_{-1}}①^{p_{-1}} + \ldots + c_{p_{-k}}①^{p_{-k}}.$$

  Then, the following record represents C

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- Finite numbers c_i are called *grossdigits*. They can be both positive and negative. Grossdigits can be expressed by several symbols in any known traditional numeral system.
- Numbers p_i are called *grosspowers*. They can be finite, infinite, and infinitesimal (the introduction of infinitesimal numbers will be given soon). They are such that $p_0 = 0$ and

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Finite numbers expressed in the new numeral system

If we have a number C such that $m = k = 0$ in the record

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then, due to the Identity property of ① ($①^0 = 1$), we have

$$C = c_0 ①^0 = c_0,$$

i.e., the number C does not contain infinite units and is equal to the grossdigit c_0 which is a conventional finite number.

Important observation

Thus, finite numbers are just a particular case in the new numeral system. Numerals having positive grosspowers represent infinite numbers and numerals having only negative grosspowers represent infinitesimal numbers.

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Numbers can contain different infinite and infinitesimal parts

- We shall call hereinafter *infinitesimals* numbers containing powers $①^i$ only with negative finite or infinite exponents $i < 0$. The simplest number from this group is $①^{-1} = \frac{1}{①}$ being the inverse element with respect to multiplication for ①:

$$\frac{1}{①} \cdot ① = ① \cdot \frac{1}{①} = 1.$$

- Note that all infinitesimals are not equal to zero. Particularly, the number $\frac{1}{①} > 0$.
- An example: the number $①^{4①} + 6①^{32.7} + 3①^{-2.1}$ is infinite, it consists of two infinite parts and one infinitesimal part.

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Application 1: Numerical differentiation

- In many practical applications it is necessary to calculate derivatives of a function $g(x)$ which is given by a computer procedure calculating its approximation $f(x)$. Since a procedure to evaluate the exact value of $f'(x)$ is usually not available, numerical approximations are used for this purpose

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}, \qquad f'(x) \approx \frac{f(x)-f(x-h)}{h},$$

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- However, round-off errors in these procedures dominate calculation when $h \to 0$. Both $f(x+h)$ and $f(x-h)$ tend to $f(x)$, so that their difference tends to the difference of two almost equal quantities and thus contains fewer and fewer significant digits.

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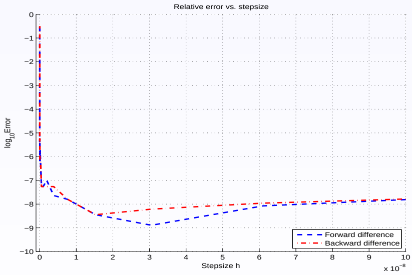
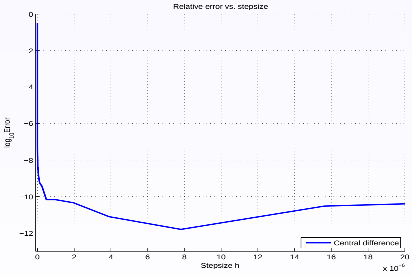
a) Forward and Backward differences



b) Central difference



Application 1: Numerical differentiation at the Infinity Computer

- Suppose that we have a computer procedure implementing the following function $g(x) = \frac{x+1}{x-1}$ and we want to evaluate the values $f(y), f'(y), f''(y)$, and $f^{(3)}(y)$ at the point $y = 3$.
- The Infinity Computer executes the following operations

$$f(3 + ①^{-1}) = (3①^0 + ①^{-1} + 1①^0)/(3①^0 + ①^{-1} - 1①^0)$$

$$= \dots$$

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- From this numeral, by applying the theorem we obtain

$$f(3) = 2, \quad f'(3) = -0.5, \quad f''(3) = 2! \cdot 0.25 = 0.5,$$

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being **exact** values of $f(x)$ and the derivatives at the point $y = 3$.

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$$f(3) = 2, \quad f'(3) = -0.5, \quad f''(3) = 2! \cdot 0.25 = 0.5,$$

$$f^{(3)}(3) = 3! \cdot (-0.125) = 0.75,$$

being **exact** values of $f(x)$ and the derivatives at the point $y = 3$.

Application 1: Numerical differentiation at the Infinity Computer

- Suppose that we have a computer procedure implementing the following function $g(x) = \frac{x+1}{x-1}$ and we want to evaluate the values $f(y), f'(y), f''(y)$, and $f^{(3)}(y)$ at the point $y = 3$.
- The Infinity Computer executes the following operations

$$f(3 + ①^{-1}) =$$ [redacted]

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Application 2

Traditional methods

Run: ...
Run: $I_0+0.001$
Run: $I_0+0.01$
Run: $I_0+0.1$
Run: I_0 does not work
Run: $I_0-0.1$
Run: $I_0-0.01$
Run: $I_0-0.001$
Run: ...

...
O_3
O_2
O_1
????
O_{-1}
O_{-2}
O_{-3}
...

Many runs and approximate results

Run: $I_0+①^{-1}$ → **The Infinity Computer** → **Exact result**

Just one run to obtain **the exact result**

Application 2: Linear systems

- Solution to the system

$$\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

is obviously given by $x_1^* = -1$, $x_2^* = 2$. It cannot be found by the method of Gauss without pivoting since already the first pivotal element $a_{11} = 0$.

- Since all the elements of the matrix are finite numbers, let us substitute the element $a_{11} = 0$ by ①$^{-1}$ and perform the Gauss transformations **without pivoting**:

$$\left[\begin{array}{cc|c} ①^{-1} & 1 & 2 \\ 2 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & ① & 2① \\ 0 & -2① + 2 & -4① + 2 \end{array}\right] \rightarrow$$

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Application 2: Linear systems

- $$\left[\begin{array}{cc|c} 1 & ① & 2① \\ 0 & 1 & \frac{-4①+2}{-2①+2} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2① - ① \cdot \frac{-4①+2}{-2①+2} \\ 0 & 1 & \frac{-4①+2}{-2①+2} \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{2①}{-2①+2} \\ 0 & 1 & \frac{-4①+2}{-2①+2} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 + \frac{1}{1-①} \\ 0 & 1 & 2 - \frac{1}{1-①} \end{array}\right].$$

- It follows immediately that the solution to the initial system

$$\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

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Application 2: Linear systems

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Application 3: Optimization with a penalty function, from De Leone and De Cosmis, 2012

- Consider the following simple 2-dimensional optimization problem with a single linear constraint (see Liuzzi, 2007)

$$\min_{x} \quad \frac{1}{2}x_1^2 + \frac{1}{6}x_2^2$$

$$\text{subject to} \quad x_1 + x_2 = 1$$

- The corresponding unconstrained optimization problem can be constructed using a penalty coefficient P as follows

$$\min_{x} \quad \frac{1}{2}x_1^2 + \frac{1}{6}x_2^2 + \frac{P}{2}(1 - x_1 - x_2)^2.$$

- Different values of P are then taken into consideration.

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- For instance, let us take $P = 20$. Then the first order optimality conditions can be written as follows

$$\begin{cases} x_1 - 20(1 - x_1 - x_2) = 0, \\ \frac{1}{3}x_2 - 20(1 - x_1 - x_2) = 0. \end{cases}$$

- Solution to this system of linear equations is the stationary point of the unconstraint problem, namely, it is

$$x_1^*(20) = \frac{20}{81}, \qquad x_2^*(20) = \frac{60}{81}$$

and it is not clear how to obtain from $(\frac{20}{81}, \frac{60}{81})$ solution to the original constrained problem.

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Application 3: The exact penalty function using ①, from De Leone and De Cosmis, 2012

- If $P = \frac{①}{2}$ then the first order optimality conditions are

$$\begin{cases} x_1 - ①(1 - x_1 - x_2) = 0, \\ \frac{1}{3}x_2 - ①(1 - x_1 - x_2) = 0. \end{cases}$$

- Therefore, the stationary point of the unconstraint problem is

$$x_1^* = \frac{1①}{1+4①}, \qquad x_2^* = \frac{3①}{1+4①},$$

$$x_1^* = \frac{1}{4} - ①^{-1}\left(\frac{1}{16} - \frac{1}{64}①^{-1} + \ldots\right), \quad x_2^* = \frac{3}{4} - ①^{-1}\left(\frac{3}{16} - \frac{3}{64}①^{-1} + \ldots\right).$$

- This means that the finite parts of x_1^* and x_2^* give us the exact solution to the original constrained problem: $\overline{x} = (\frac{1}{4}, \frac{3}{4})$.

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Application 4: solving ODEs

- We want to approximate the solution $y(x)$, $x \in [a,b]$, of the initial value problem (also called the Cauchy problem) for an ordinary differential equation (ODE)

$$y'(x) = f(x,y), \qquad y(x_0) = y_0, \qquad x_0 = a, \tag{1}$$

where a and b are finite numbers and $y(x_0) = y_0$ is called the initial condition.

- We suppose that $f(x,y)$ is given by a computer procedure. Since very often in scientific and technical applications it can happen that the person who wants to solve (1) is not the person who has written the code for $f(x,y)$, **we suppose that the person solving (1) does not know the structure of $f(x,y)$, i.e., it is a black box for him/her.**

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Application 4: Traditional computers use a finite approximation step h

- In the literature, there exist numerous numerical algorithms constructing a sequence $y_1, y_2, y_3, \ldots$ approximating the exact values $y(x_1), y(x_2), y(x_3), \ldots$ that the solution $y(x)$ of $y'(x) = f(x, y)$ assumes at points $x_1, x_2, x_3, \ldots$ starting from an initial value $y_0 = y(x_0)$.
- To produce approximations $y_1, y_2, y_3, \ldots$, traditional computers work with finite values of the approximation step h introducing so errors at each step of approximation. In order to obtain more accurate approximations, it is necessary to decrease the step h increasing so the number of steps of the method (the computations become more expensive). In any case, h always remains finite.

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An example

- Let us find $y''(0)$ for the problem

$$y'(x) = \frac{y - 2xy^2}{1+x}, \qquad y(0) = y_0 = 0.4. \tag{2}$$

Then the exact $y''(0) = -0.32$ since the exact solution of (2) is

$$y(x) = \frac{1+x}{2.5+x^2}.$$

- In order to find $y''(0)$ we start by calculating y_1, y_2 and $\frac{\triangle^2_{①^{-1}}}{①^{-2}}$:

$$y_1 = 0.4 + ①^{-1} f(0, 0.4) = 0.4 + 0.4①^{-1},$$

$$y_2 = y_1 + ①^{-1} f(①^{-1}, y_1) = 0.4 + 0.8①^{-1} - 0.32①^{-2} - 0.32①^{-3}.$$

$$\frac{\triangle^2_{①^{-1}}}{①^{-2}} = \frac{y_0 - 2y_1 + y_2}{①^{-2}} = \frac{-0.32①^{-2} - 0.32①^{-3}}{①^{-2}} = -0.32 - 0.32①^{-1} = y''(0) + O(①^{-1}),$$

and we have $y''(0) = -0.32$.

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and we have $y''(0) = -0.32$.

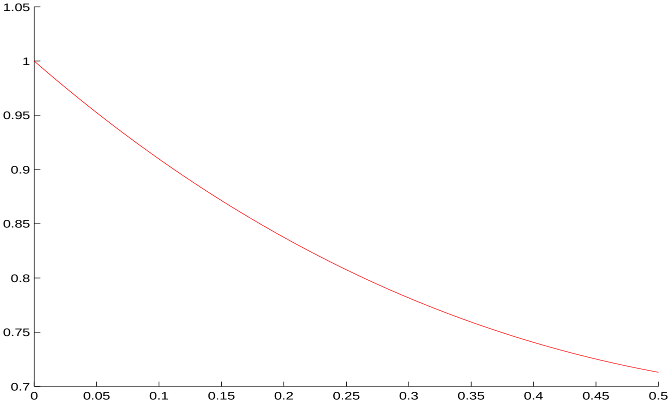
ε_RK4 and ε_TIC are relative errors at $x = 0.2$, N_TIC is the number of evaluation of $f(x, y)$ executed by a method to obtain the accuracy ε_TIC at the point $x = 0.2$. The RK4 method executes 20 evaluations for each test problem.

	ε_RK4	ε_TIC	N_TIC
1	-8.625380e-09	-5.916870e-09	6
2	8.111570e-09	4.191510e-09	6
3	4.126850e-09	2.132480e-09	6
4	3.898340e-08	2.132480e-09	6
5	1.277260e-07	1.136930e-08	7
6	-5.965290e-04	-3.244200e-04	10
7	-8.164050e-05	5.845400e-05	9
8	-8.182930e-05	5.858170e-05	9
9	-5.788030e-09	-4.082110e-09	10
10	-1.769490e-09	7.941280e-11	7
11	8.985770e-09	4.096000e-09	11
12	2.957750e-10	-1.607820e-10	10

ε_RK4 and ε_TIC are relative errors at $x = 1.0$, N_1.0 is the number of evaluation of $f(x, y)$ executed by the TIC method. RK4 method executes 100 evaluations for each problem.

	ε_RK4	ε_TIC	N_TIC
1	-2.205680e-08	-1.513060e-08	30
2	3.264290e-08	1.686770e-08	30
3	2.063430e-08	1.066240e-08	30
4	3.025460e-07	1.654990e-08	30
5	6.385330e-07	5.660170e-08	35
6	-2.986200e-03	-1.623150e-03	50
7	-1.224800e-06	8.764000e-07	45
8	-1.346740e-06	9.472220e-07	45
9	-7.468060e-09	-8.006580e-10	50
10	6.859090e-08	-3.028460e-10	35
11	3.821950e-08	1.379340e-09	55
12	7.691030e-09	-2.016510e-11	50

The possibility to improve the accuracy of a solution going forward and backward, $k = 2$



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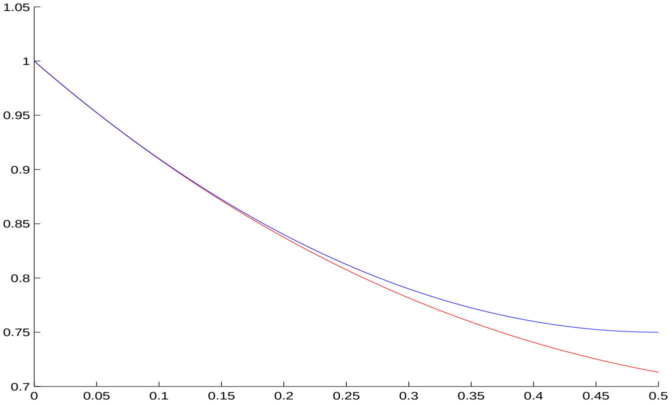
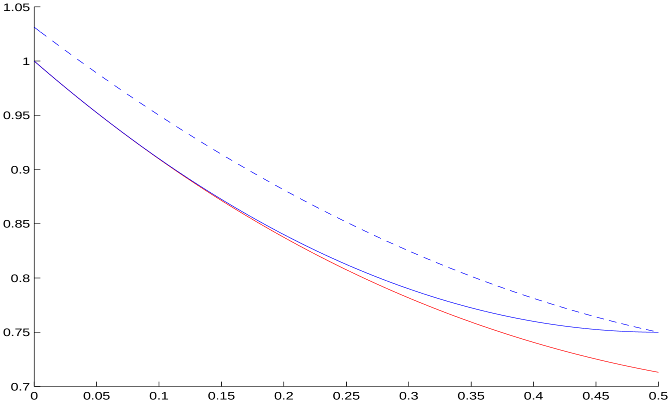


The possibility to improve the accuracy of a solution going forward and backward, $k = 2$



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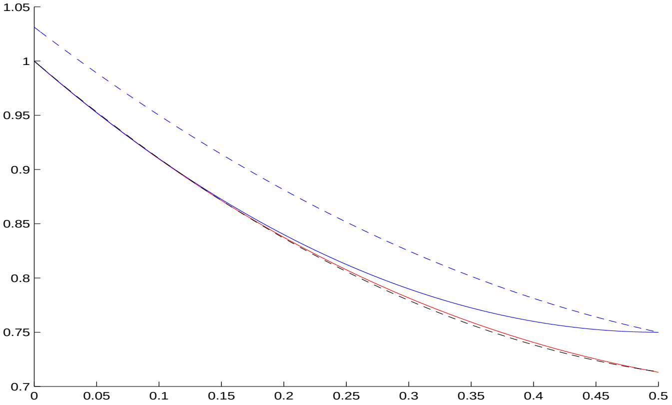
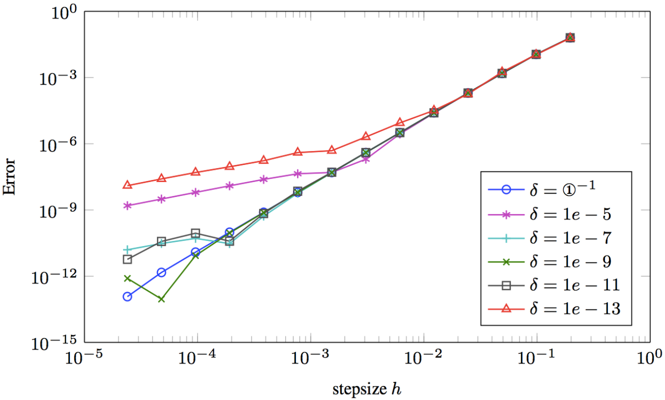


Errors generated by a forward-backward method with $k = 2$ for different choices of the parameter δ to approximate the required derivatives of f



It becomes easier to work with fractals

For example, the total length, $L(n)$, of intervals forming Cantor's set after n steps is $L(n) = 2^n \frac{1}{3^n}$ (2^n equal intervals having the length $\frac{1}{3^n}$ each). Thus, after $n =$ ① steps $L(n)$ is the following infinitesimal number $L(①) = 2^{①} \cdot \frac{1}{3^{①}} = (\frac{2}{3})^{①}$.

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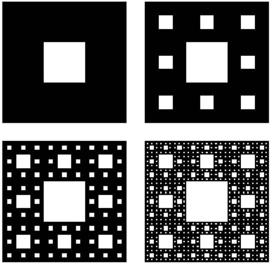
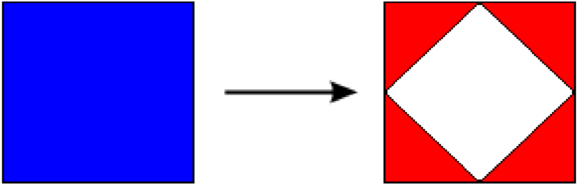
'Infinity Computer' Calculates Area Of Sierpinski Carpet Exactly

Mathematicians have never been comfortable handling infinities, such as those that crop up in the area of a Sierpinski carpet. But an entirely new type of mathematics looks set to by-pass the problem

By KFC



Blinking fractals – objects with several alternating each other fractal mechanisms



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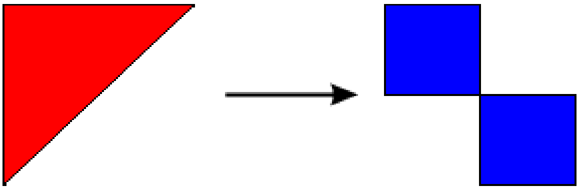
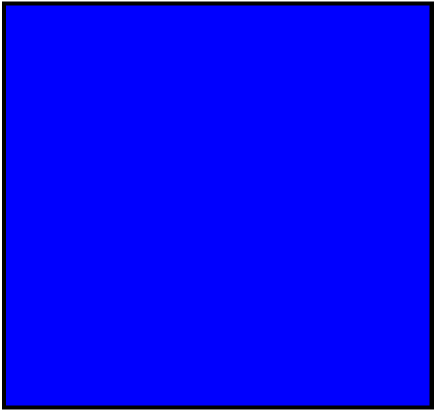
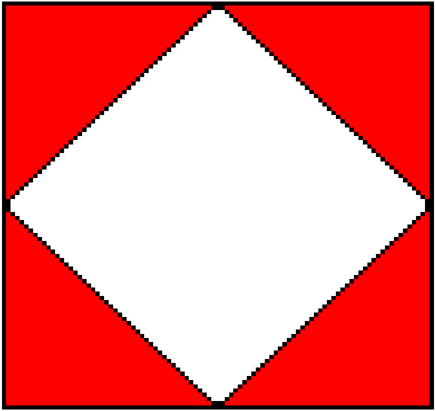


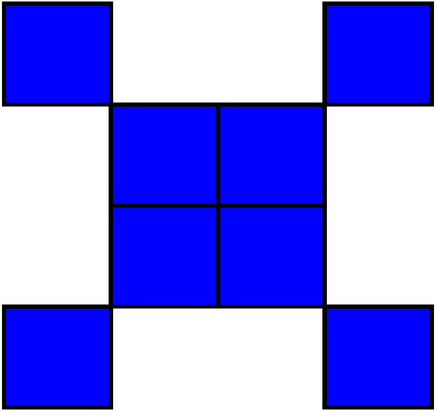
Iteration 0



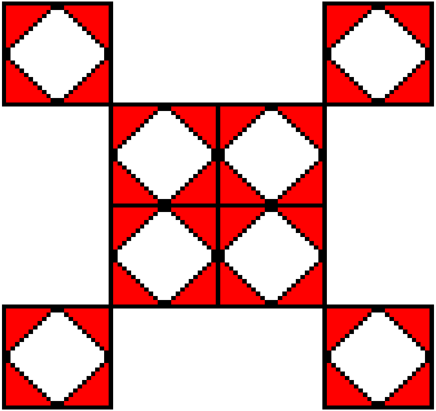
Iteration 1



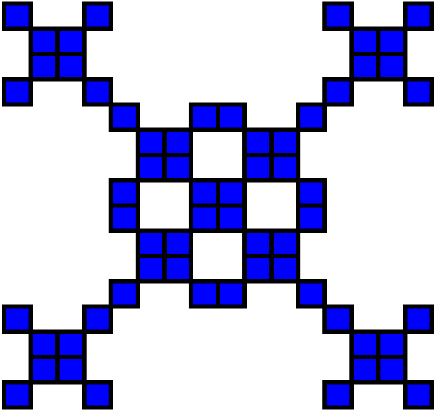
Iteration 2



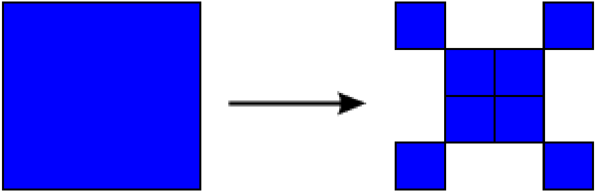
Iteration 3



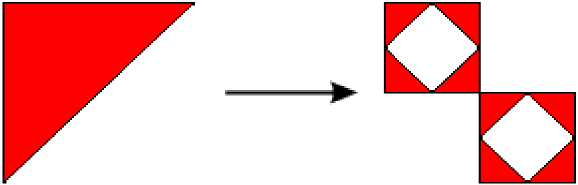
Iteration 4



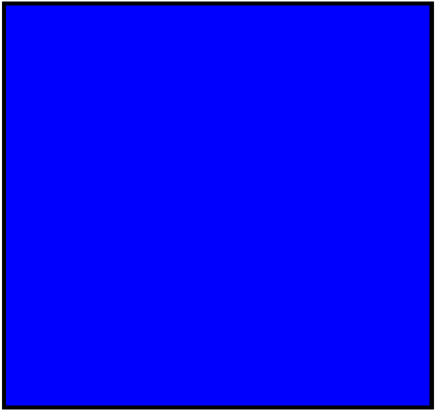
The first traditional fractal mechanism regarding blue squares separated from the blinking process



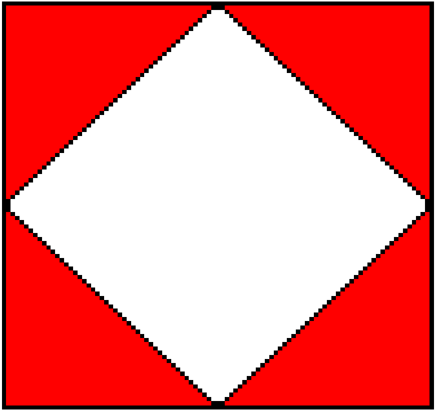
The second traditional fractal mechanism regarding red triangles separated from the blinking process



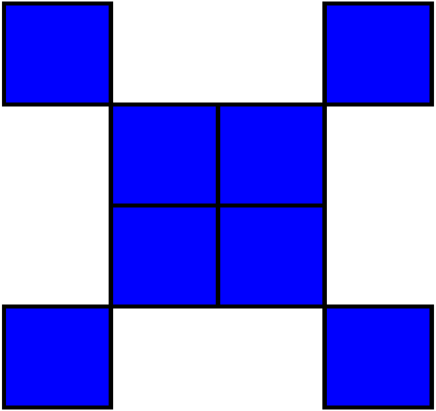
Iteration 0



Iteration 1



Iteration 2



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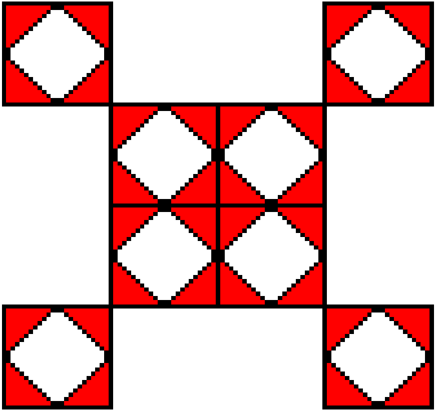
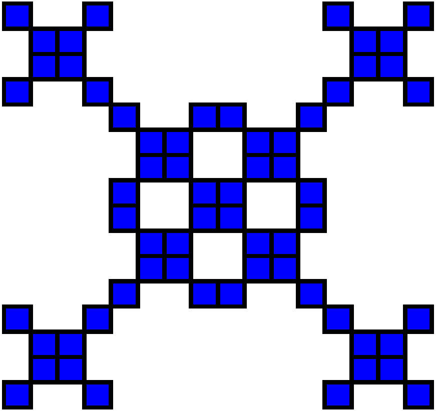


Iteration 4



Where to read about the Infinity Computer?

- *Basic ideas presented in the talk were described in a simple and informal way in the book* **Yaroslav Sergeyev, Arithmetic of infinity**.
- *Numerous papers in international journals written by several authors and a lot of an additional information regarding the the new methodology, the Infinity Computer, and the patents can be found at*

  http://www.theinfinitycomputer.com
- *This site contains videos of lectures, as well.*

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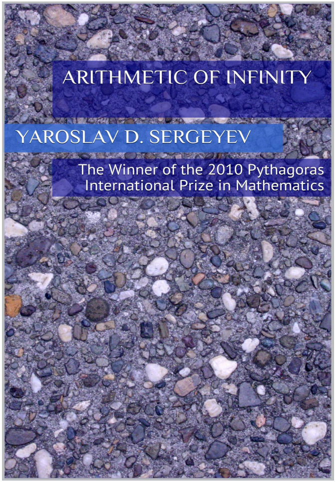
This popular e-book is available in Amazon



Opinions of experts in international scientific journals

- "These ideas and future hardware prototypes may be productive in all fields of science where infinite and infinitesimal numbers (derivatives, integrals, series, fractals) are used." A. Adamatzky, Editor-in-Chief of the **International Journal of Unconventional Computing**.
- "By introducing a new infinite unit ... he shows that it is possible to effectively work with infinite and infinitesimal quantities and to solve many problems connected to them in the field of applied and theoretical mathematics." R. De Leone, President of the Operations Research Society of Italy (2007-2012), **Applied Mathematics and Computation**.
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